

## ON OVERTWISTED, RIGHT-VEERING OPEN BOOKS

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ABSTRACT. We exhibit infinitely many overtwisted, right-veering, non-destabilizable open books, thus providing infinitely many counterexamples to a conjecture of Honda–Kazez–Matić. The page of all our open books is a four-holed sphere and the underlying 3-manifolds are lens spaces.

## 1. INTRODUCTION

The purpose of this note is to construct infinitely many counterexamples to a conjecture of Honda, Kazez and Matić from [12]. For the basic notions of contact topology not recalled below we refer the reader to [4, 6].

Let  $S$  be a compact, oriented surface with boundary and  $\text{Map}(S, \partial S)$  the group of orientation-preserving diffeomorphisms of  $S$  which restrict to  $\partial S$  as the identity, up to isotopies fixing  $\partial S$  pointwise. An *open book* (a.k.a. an *abstract open book*) is a pair  $(S, \Phi)$  where  $S$  is a surface as above and  $\Phi \in \text{Map}(S, \partial S)$ . Giroux [8] introduced a fundamental operation of *stabilization*  $(S, \Phi) \rightarrow (S', \Phi')$  on open books, and proved the existence of a 1–1 correspondence between the set of open books modulo stabilization and the set of contact 3-manifolds modulo isomorphism (see e.g. [5] for details). Honda, Kazez and Matić [11] showed that a contact 3-manifold is tight if and only if it corresponds to an equivalence class of open books  $(S, \Phi)$  all of whose monodromies  $\Phi$  are *right-veering* (in the sense of [11, Section 2]). In [9, 11] it is also showed that every open book can be made right-veering after a sequence of stabilizations. In [12], Honda, Kazez and Matić proved that, when  $S$  is a holed torus, the contact structure corresponding to  $(S, \Phi)$  is tight if and only if  $\Phi$  is right-veering, and conjectured that a non-destabilizable right-veering open book corresponds to a tight contact 3-manifold. The Honda–Kazez–Matić conjecture was recently disproved by Lekili [13], who produced a counterexample  $(S, \Phi)$  with  $S$  equal to a four-holed sphere and whose underlying 3-manifold is the Poincaré homology sphere.

We shall now describe our examples. Denote by  $\delta_\gamma \in \text{Map}(S, \partial S)$  the class of a positive Dehn twist along a simple closed curve  $\gamma \subset S$ .

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**Theorem 1.1.** *Let  $S$  be an oriented four-holed sphere, and  $a, b, c, d, e$  the simple closed curves on  $S$  shown in Figure 1. Let  $h, k \geq 1$  be integers.*

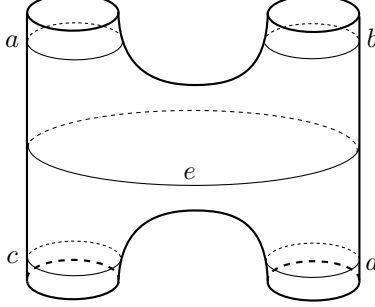


FIGURE 1. The four-holed sphere  $S$

Define

$$\Phi_{h,k} := \delta_a^h \delta_b \delta_c \delta_d \delta_e^{-k-1} \in \text{Map}(S, \partial S).$$

Then,

- The underlying 3-manifold  $Y_{(S, \Phi_{h,k})}$  is the lens space

$$L((h+1)(2k-1) + 2, (h+1)k + 1);$$

- the associated contact structure  $\xi_{(S, \Phi_{h,k})}$  is overtwisted;
- $\Phi_{h,k}$  is right-veering;
- $(S, \Phi_{h,k})$  is not destabilizable.

Warning: in the above statement we adopt the convention that the lens space  $L(p, q)$  is the oriented 3-manifold obtained by performing a rational surgery along an unknot in  $S^3$  with coefficient  $-p/q$ .

We prove Theorem 1.1 in Section 2. The proof can be outlined as follows. In Proposition 2.1 we use elementary arguments to determine a contact surgery presentation for the contact 3-manifold  $(Y_{(S, \Phi_{h,k})}, \xi_{(S, \Phi_{h,k})})$ , and in Corollary 2.2 we apply Proposition 2.1 and a few Kirby calculus moves to identify the underlying 3-manifold  $Y_{(S, \Phi_{h,k})}$ . In Proposition 2.3 we appeal to calculations from [13] to deduce that the contact Ozsváth–Szabó invariant of  $\xi_{(S, \Phi_{h,k})}$  vanishes, and we conclude from the fact that  $Y_{(S, \Phi_{h,k})}$  is a lens space that  $\xi_{(S, \Phi_{h,k})}$  must be overtwisted. We show that  $\Phi_{h,k}$  is right-veering in Lemma 2.4 by observing that this result follows directly from [2, Theorem 4.3], but can also be deduced imitating the proof of [13, Theorem 1.2], i.e. applying [11, Corollary 3.4]. Finally, we use results from [1, 13] to conclude that  $(S, \Phi_{h,k})$  is not destabilizable.

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## 2. PROOF OF THEOREM 1.1

Recall that every contact structure has a *contact surgery presentation* [3]. We refer the reader to [3] for the basic properties of contact surgeries, and to [14] for the use of the ‘front notation’ in contact surgery presentations, in particular for the meaning of Figure 2 below.

**Proposition 2.1.** *For  $h, k \geq 1$ , the contact structure  $\xi_{(S, \Phi_{h,k})}$  has the contact surgery presentation given by Figure 2.*

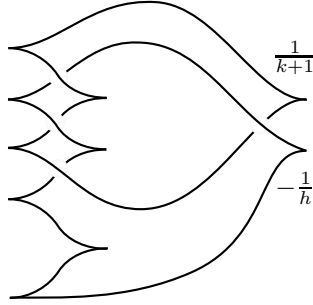


FIGURE 2. Contact surgery presentation for  $\xi_{(S, \Phi_{h,k})}$ ,  $h, k \geq 1$ .

*Proof.* Figure 3(a) represents an open book  $(A, f)$ , where  $A$  is an annulus and  $f$  is a positive Dehn twist along the core of  $A$ . The associated contact 3-manifold is the standard contact 3-sphere  $(S^3, \xi_{\text{st}})$ , the annulus  $A$  can be viewed as the page of an open book decomposition of  $S^3$ , and the curve  $\kappa$  in the picture can be made Legendrian via an isotopy of the contact structure, in such a way that the contact framing on  $\kappa$  coincides with the framing induced on it by the page (see e.g. [5, Figure 11]). The knot  $\kappa$  is the unique Legendrian unknot in  $(S^3, \xi_{\text{st}})$  having Thurston–Bennequin invariant  $\text{tb}(\kappa) = -1$  and rotation number  $\text{rot}(\kappa) = 0$ . A suitable choice of orientation for  $\kappa$  uniquely specifies its *negative* oriented Legendrian stabilization  $\kappa_-$ , which satisfies  $\text{tb}(\kappa_-) = -2$  and  $\text{rot}(\kappa_-) = -1$ . As shown in [5],  $\kappa_-$  can be realized as sitting on the page of a Giroux stabilization  $(A', f')$  of  $(A, f)$ . This is illustrated in Figure 3(b), assuming the orientation on  $\kappa$  was taken to be “counterclockwise” in Figure 3(a). Finally, Figure 3(c) shows an open book  $(S, f'')$  obtained by Giroux stabilizing  $(A', f')$  and containing both  $\kappa_-$  and  $(\kappa_-)_-$  in  $S$  ( $\kappa_-$  was also given the “counterclockwise” orientation in Figure 3(b)). Clearly  $(S, f'')$  still corresponds to  $(S^3, \xi_{\text{st}})$ , and it is well-known that  $\kappa_-$ ,  $(\kappa_-)_-$  are the two Legendrian knots illustrated in Figure 2 (when oriented “clockwise” in that picture). By definition,  $\Phi_{h,k}$  is obtained by pre-composing  $f''$  with  $k+1$  negative Dehn twists along parallel copies of  $\kappa_-$  and  $h$  positive Dehn twists along parallel copies of  $(\kappa_-)_-$ . Moreover, if  $m \neq 0$  is an integer,  $\frac{1}{m}$ -contact surgery along any Legendrian knot  $\lambda$  is equivalent to  $\frac{m}{|m|}$ -contact surgeries along  $|m|$  Legendrian push-offs of  $\lambda$  [3].

Since page and contact framings coincide and by e.g. [5, Theorem 5.7] positive (negative, respectively) Dehn twists correspond to  $-1$ -contact surgeries ( $+1$ -contact surgeries, respectively), it is easy to check that the resulting contact structure is given by the contact surgery presentation of Figure 2.  $\square$

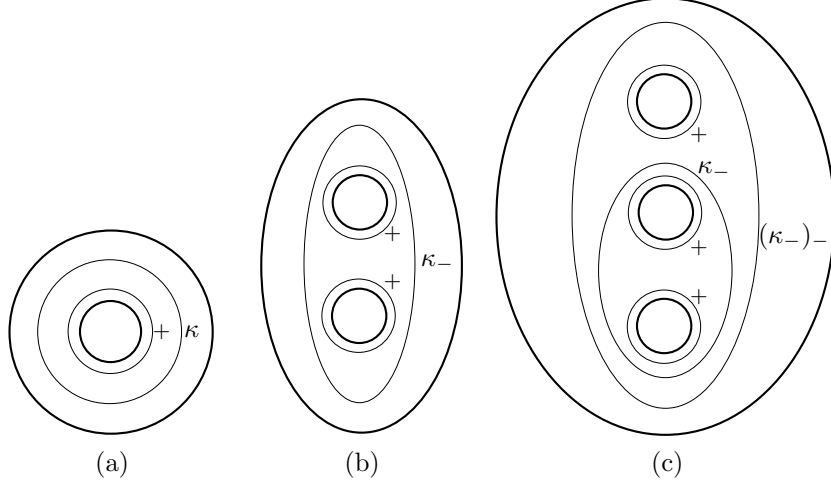


FIGURE 3. Determination of the contact surgery presentation.

**Corollary 2.2.** *For  $h, k \geq 1$ , the oriented 3-manifold underlying the open book  $(S, \Phi_{h,k})$  is the lens space  $L((h+1)(2k-1)+2, (h+1)k+1)$ .*

*Proof.* Using the fact that the two Legendrian unknots illustrated in Figure 2 have Thurston–Bennequin invariants  $-2$  and  $-3$ , it is easy to check that the topological surgery underlying Figure 2 is given by the first (upper left) picture of Figure 4. Two  $+1$ -blowups and two inverse slam-dunks give

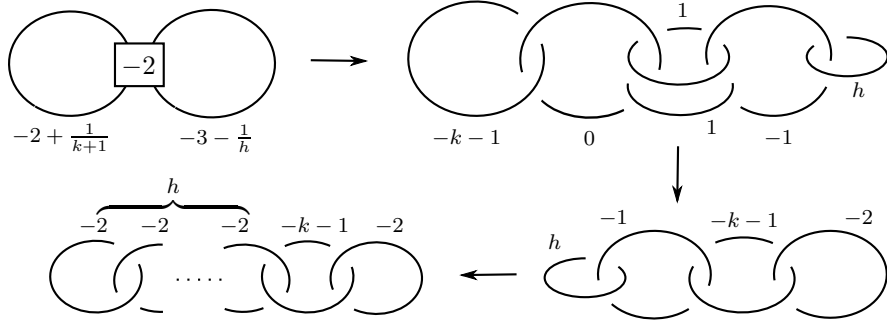


FIGURE 4. Determination of the underlying 3-manifold.

the second picture, while the third picture is obtained from the second one by sliding the  $-1$ -framed knot over the  $0$ -framed knot and then applying

two  $+1$ -blow-downs. The last picture is obtained simply converting the  $h$ -framed unknot in the third picture into the string of  $-2$ -framed unknots via a sequence of  $-1$ -blowups and a final  $+1$ -blowdown. The last picture shows that the underlying 3-manifold  $Y_{(S, \Phi_{h,k})}$  is obtained by performing a rational surgery on an unknot in  $S^3$  with coefficient  $-p/q$ , where

$$\frac{p}{q} = 2 - \frac{1}{k+1 - \frac{1}{2 - \frac{1}{\ddots - \frac{1}{2}}}} = \frac{(h+1)(2k-1)+2}{(h+1)k+1}.$$

Therefore, according to our conventions  $Y_{(S, \Phi_{h,k})}$  can be identified with the lens space  $L((h+1)(2k-1)+2, (h+1)k+1)$ .  $\square$

**Proposition 2.3.** *For  $h, k \geq 1$ , the contact structure  $\xi_{(S, \Phi_{h,k})}$  is overtwisted.*

*Proof.* By [7, 10] a contact structure on a lens space is either overtwisted or Stein fillable. Moreover, Stein fillable contact structures have non-zero contact Ozsváth–Szabó invariant [15]. Finally, [13, Theorem 1.3] immediately implies that the contact invariant of  $(S, \Phi_{h,k})$  vanishes, therefore  $\xi_{(S, \Phi_{h,k})}$  must be overtwisted.  $\square$

**Lemma 2.4.** *For  $h, k \geq 1$ , the diffeomorphism class*

$$\Phi_{h,k} = \delta_a^h \delta_b \delta_c \delta_d \delta_e^{-k-1} \in \text{Map}(S, \partial S)$$

*is right-veering.*

*Proof.* The lemma follows immediately from the statement of [2, Theorem 4.3]. Alternatively, one can imitate the proof of [13, Theorem 1.2]. Indeed, applying [11, Corollary 3.4] to the monodromy  $\Phi_1 = \delta_e^{-k-1}$  and a properly embedded arc  $\gamma_{cd} \subset S$  disjoint from the curve  $e$  and connecting the components  $\partial_c$  and  $\partial_d$  of  $\partial S$  parallel to the curves  $c$  and  $d$  shows that  $\Phi_2 = \delta_d \delta_e^{-k-1}$  is right-veering with respect to  $\partial_d$ . Another application of the corollary to  $\Phi_2$  and  $\gamma_{cd}$  shows that  $\Phi_3 = \delta_c \delta_d \delta_e^{-k-1}$  is right-veering with respect to  $\partial_c$ . Moreover, since  $\delta_c$  is right-veering with respect to  $\partial_c$  and the composition of right-veering diffeomorphisms is still right-veering [11],  $\Phi_3$  is right-veering with respect to  $\partial_d$  as well. Applying the corollary in the same way to  $\Phi_3$  and an arc connecting the components of  $\partial S$  parallel to the curves  $a$  and  $b$  yields the statement of the lemma.  $\square$

*Proof of Theorem 1.1.* Corollary 2.2, Proposition 2.3 and Lemma 2.4 establish the first three portions of the statement. We are only left to show that  $(S, \Phi_{h,k})$  is not destabilizable for every  $h, k \geq 1$ . If  $(S, \Phi_{h,k})$  were destabilizable, it would be a stabilization of an open book  $(S', \Phi')$ , with  $S'$  a three-holed sphere and  $\Phi' = \tau_1^{a_1} \tau_2^{a_2} \tau_3^{a_3}$ , where  $a_i \in \mathbb{Z}$  and  $\tau_i$  is a positive Dehn twist along a simple closed curve parallel to the  $i$ -th boundary components of  $S'$ ,  $i = 1, 2, 3$ . By [1, Theorem 1.2],  $\xi_{(S, \Phi_{h,k})}$  is tight if and only

if  $a_i \geq 0$ ,  $i = 1, 2, 3$ . Therefore, by Proposition 2.3 at least one of these exponents must be strictly negative. But the proof of [13, Theorem 1.2] shows that when one of the  $a_i$ 's is negative, any stabilization of  $(S', \Phi')$  to an open book with page a four-holed sphere is not right-veering. This would contradict Lemma 2.4, therefore we conclude that  $(S, \Phi_{h,k})$  cannot be destabilizable.  $\square$

*Note:* after completing the first version of this paper the author was informed of independent, unpublished work of A. Wand containing, in particular, a different proof of Proposition 2.3.

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